Lecture 4

Gaussian Elimination

The basic idea behind the Gaussian elimination method is to transform general linear system of equations $Ax = b$ into triangular form. To do this we need to replace selected nonzero entries of matrix by zeros. This can be accomplished by taking linear combinations of rows.

Let us first see how Gaussian elimination works. Consider the following system of equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ 2x_1 - 2x_2 + x_3 &= 1 \\ x_1 + x_2 - x_3 &= 0 \end{aligned}$$

The matrix representation of this system is

$$Ax = b \Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 2 & -2 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 0 \end{pmatrix}$$

Now there are several operations that one can perform on a system of equations, without changing its solution.

1. We can replace any equation by a non-zero constant times the original equation
2. We can replace any equation by its sum with another equation
3. We can carry out the above operations any number of times.

In the matrix language the operations we apply on augmented matrix

$$[A][b] = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -2 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}$$

Thus for example we can add –2 times the first row to the second row and –1 times the first row to the third row to obtain the following equivalent system

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -4 & -1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 6 \\ -11 \\ -6 \end{bmatrix}$$

Which being an upper triangular system, is readily solved by back substitution.

LU Factorization

Gaussian elimination can be used for LU factorization. Let's carry out an example. Consider the following matrix (Note: the operations below are applied only to the unaugmented matrix ***A*** of a linear system ***Ax*** **=** ***b***)

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 5 & 5 \\ 0 & 3 & 18 & 19 \end{bmatrix}$$

Let's associate with this stage a column vector

$$c_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \\ -1 \end{bmatrix}$$

Here the first non-zero component 1 indicates that the first row unchanged, the second component indicates that we multiplied the first row by –2 before subtracting it from the second row, third component indicates that we multiplied the first row by 2 and subtracting it from the third row, and the fourth component indicates that we multiplied the first row by 1 and subtracted it from the fourth row.

In the next stage we ignore row one, leave row 2 unchanged, and add multiples of row 2 to rows 3 and 4. The following column vector can thus characterize this stage

$$c_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

and subsequent matrix will be

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 12 & 10 \end{bmatrix}$$

In the next stage we ignore the first two rows, leave row 3 unchanged, and multiply row 3 by 2/3 and add it to the last row. The corresponding column vector will be

$$c_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \end{bmatrix}$$

and the resulting matrix will be

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Let us write down one last colun vector to indicate that there's nothing left to do

$$c_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Now set U denote the upper triangular matrix that represents the result of the final stage of Gaussian elimination

$$U = \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

and let L the lower triangular matrix that's formed by adjoining the column vectors c_1, c_2, c_3, c_4:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & 3 & 4 & 1 \end{bmatrix}$$

Then, surprise, these two matrices provide an *LU* factorization of the original matrix ***A.***

New and Improved Notation

In the preceding example, we saw that we could simultaneously identify an LU factorization of a matrix A, if we conscientiously kept track of the multipliers of the pivot rows. We did this by associating a column vector to each stage of the Gaussian elimination procedure. What I now want to describe a notational hat-trick that allows us to work only with the matrix A. The idea is this: we keep track of the multipliers by replacing the multipliers for a given row, say the i^{th} row for the k^{th} stage of Gaussian elimination in the ***ki*** slot of the matrix. Of course, these new entries are not realy components of the matrix A, we can put them there though with out losing information because after the k^{th} stage of Gaussian elimination, the matrix A will always have a zero in the ***ki*** slot. To make the special interpretation of these entries manifest we will underline them. The example above would them work out as follows:

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -3 & 6 & 11 \\ -12 & 5 & 1 & -3 \\ 6 & 1 & 20 & 23 \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 & 4 \\ \underline{2} & 1 & 2 & 3 \\ \underline{-2} & 1 & 5 & 5 \\ \underline{1} & 3 & 18 & 19 \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 & 4 \\ \underline{2} & 1 & 2 & 3 \\ \underline{-2} & \underline{1} & 3 & 2 \\ \underline{1} & \underline{3} & 12 & 10 \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 & 4 \\ \underline{2} & 1 & 2 & 3 \\ \underline{-2} & \underline{1} & 3 & 2 \\ \underline{1} & \underline{3} & \underline{4} & 2 \end{bmatrix}$$

Now the *LU* factorization of ***A*** can be obtained by pulling off the underline entries into a unit lower triangular matrix ***L*** and interpreting the entries that remain as the components of a upper triangular matrix ***U***.

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \underline{2} & 1 & 0 & 0 \\ \underline{-2} & \underline{1} & 1 & 0 \\ \underline{1} & \underline{3} & \underline{4} & 1 \end{bmatrix} \quad U = \begin{bmatrix} 6 & -2 & 2 & 4 \\ \underline{0} & 1 & 2 & 3 \\ \underline{0} & \underline{0} & 3 & 2 \\ \underline{0} & \underline{0} & \underline{0} & 2 \end{bmatrix}$$

Thus, another method of solution for a linear system of the form ***Ax* = *b*** would be to carry out the technique above to find LU factorization of the unaugmented matrix ***A*** and then solve ***Lz* = *b*** for ***z*** and finally ***Ux* = *z***.